Final Exam : MTH 111, Spring 2016

Ayman Badawi

~~100~~ 84

- Make sure you have 5 different pages.
- Throughout the exam, write your solution clearly. Otherwise points will be deducted.
- Mobiles are not allowed in this exam.

(28 items, each item = 3points, total = 84 points)

$\boxed{Q}$ **1.** (i) Let $f(x) = -x^3 + 6x^2 + 10$. Then

a. For what values of x does $f(x)$ increase?

$f'(x) = -3x^2 + 12x = 0 \Rightarrow -3x(x-4) = 0 \Rightarrow x=0,\ x=4$

x		0		4	
y'	$-$	0	$(+)$	0	$-$
y	$\searrow$		$\nearrow$		$\searrow$

$f(x)$ increase in interval ~~of~~ $(0, 4)$

b. For what values of x does $f(x)$ have a maximum value?

$f''(x) = -6x + 12 \Rightarrow f''(0) = 0 + 12 = 12$ Min

$f''(4) = -6(4) + 12 = -12 < 0 \Rightarrow$ for $\boxed{x = 4}$ it $f(x)$ is Max

$\boxed{Q}$ **2.** (i) Given $f'(1) = 7$ and $y = f(x^2 + x - 5)$. Then $y'(2) =$

$y' = [f'(x^2 + x - 5)](2x+1) \Rightarrow y'(2) = [f'(4 + 2 - 5)](2(2)+1)$

$y'(2) = 7 \times 5 = 35$

(ii) Let $f(x) = -3e^{(2x^2 - x - 1)}$. Then $f'(1) =$

$f'(x) = -3(4x-1)e^{(2(x)^2 - x - 1)} \Rightarrow f'(1) = -3(3)e^{(2-1-1)} = -9$

(iii) Let $f(x) = ln\left((5x-9)^3(2x-3)^7\right)$. The the slope of the tangent line at $x = 2$ is

$f(x) = 3\ln(5x-9) + 7\ln(2x-3)$

$f'(x) = \frac{3(5)}{5x-9} + \frac{7(2)}{2x-3} = \frac{15}{5x-9} + \frac{14}{2x-3}$

$M = f'(2) = \frac{15}{5(2)-9} + \frac{14}{2(2)-3} = \frac{15}{1} + \frac{14}{1} = 29$

Q 3. (i) $\int \frac{x}{x^2+3}\,dx = \int x(x^2+3)^{-1}\,dx = \frac{1}{2}\int 2x(x^2+3)^{-1}\,dx = \frac{1}{2}\ln(|x^2+3|)+$

$k(x) = x^2+3 \rightarrow k'(x) = 2x$

(ii) $\int \frac{e^x+1}{(e^x+x+1)^2}\,dx = \int (e^x+1)(e^x+x+1)^{-2}\,dx = \frac{(e^x+x+1)^{-1}}{-1}+c$

$= \frac{-1}{(e^x+x+1)} + c$

$k(x) = e^x+x+1 \Rightarrow k'(x) = e^x+1$

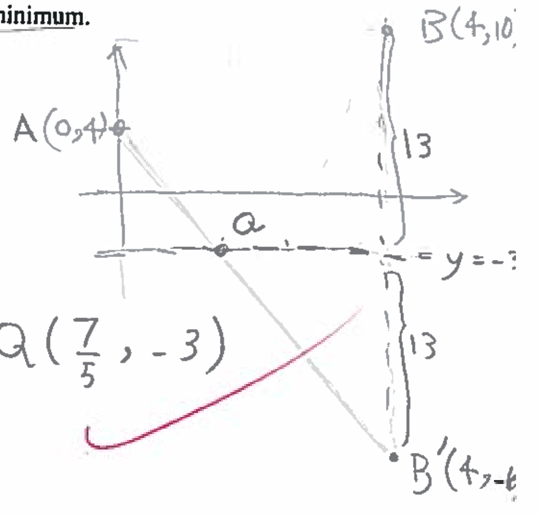
(iii) $\int x^3(x+1)^2\,dx = \int x^3(x^2+2x+1)\,dx = \int (x^5+2x^4+x^3)\,dx$

$= \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + c$

(iv) $\int 10(5x+7)^9\,dx = \frac{\cancel{10}^{2}}{\cancel{5}}\int 5(5x+7)^9\,dx = \frac{\cancel{2}(5x+7)^{10}}{\cancel{10}} + c$

$k(x) = 5x+7 \longrightarrow k'(x) = 5$ $\qquad = \frac{(5x+7)^{10}}{5} + c$

Q 4. Let $A = (0,4)$, $B = (4,10)$. Find a point Q on the line $y = -3$ such that $|BQ| + |QA|$ is minimum.



$B'(4, -3-13) = (4, -16)$

$m = \frac{4+16}{0-4} = \frac{20}{-4} = -5$

$\xrightarrow{(0,4)}$ $y-4 = -5(x-0) \Rightarrow y = -5x+4$

$\xrightarrow{y=-3}$ $-3 = -5x+4 \Rightarrow -7 = -5x \Rightarrow x = \frac{7}{5} \Rightarrow Q(\frac{7}{5}, -3)$

Q 5. Find two positive integers x, y such that $xy = 9$ and $x+4y$ is minimum

$xy = 9 \Rightarrow y = \frac{9}{x}$

$S = x+4y = x+4(\frac{9}{x}) = x+36x^{-1}$

Positive

$\Rightarrow S' = 1-36x^{-2} = 0 \Rightarrow 1 = \frac{36}{x^2} \Rightarrow x^2 = 36 \Rightarrow x = \pm 6 \Rightarrow \boxed{x=6}$

$S'' = 72x^{-2} \Rightarrow S''(6) = \frac{72}{36} = 2 > 0 \Rightarrow$ if $x=6 \Rightarrow S$ will be Min.

$y = \frac{9}{x} = \frac{9}{6} = \frac{3}{2} = 1.5 \Longrightarrow S_{min} = x+4y = 6+6 = 12$

Q 6. Find the area of the largest rectangle that you can construct between $y = x + 4$, $y = -x + 4$ and the $x - axis$. Note that such rectangle is symmetric about the $y - axis$. See picture.

$a = x$

$b = -a + 4$

$S = (2a)\,b$

$S = (2a)(-a + 4)$

$S = (-2a^2 + 8a)$

$S' = -4a + 8 = 0 \Rightarrow 4a = 8$

$\Rightarrow a = 2$

$S'' = -4 \Rightarrow S''(2) = -4 < 0$

$\Rightarrow$ S is max when $\boxed{a = 2}$

$\Rightarrow b = -a + 4 = -2 + 4 = 2$

$\Rightarrow S_{Max} = 2a.b = 2(2) \times 2 = 8$

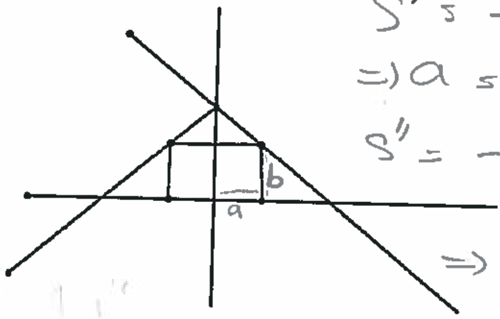


Figure 1. Largest rectangle between $y = x + 4$, $y = -x + 4$ and the $x - axis$.

Q 7. Find the area of the region bounded by $y = x - 3$ and the $x - axis$, where x is between 0 and 4 (see picture).

$y = x - 3 = 0 \Rightarrow x = 3$

$\left|\int_0^3 x - 3\right| + \int_3^4 x - 3 \Rightarrow A = \left|\frac{x^2}{2} - 3x\right|_0^3 + \frac{x^2}{2} - 3x\Big|_3^4 = \left|\left(\frac{9}{2} - 9\right) - 0\right| + \left(\frac{16}{2} - 12\right) - \left(\frac{9}{2} - 9\right)$

$A = 4.5 + 0.5 = 5$

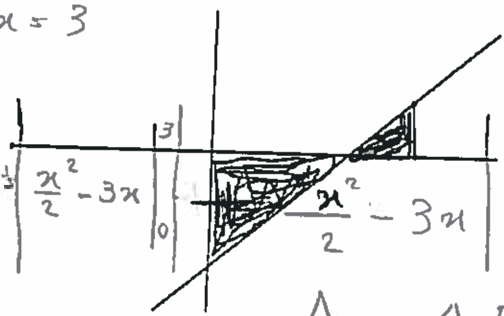


Figure 2. area of the region bounded by $y = x - 3$ and the $x - axis$, where x is between 0 and 4.

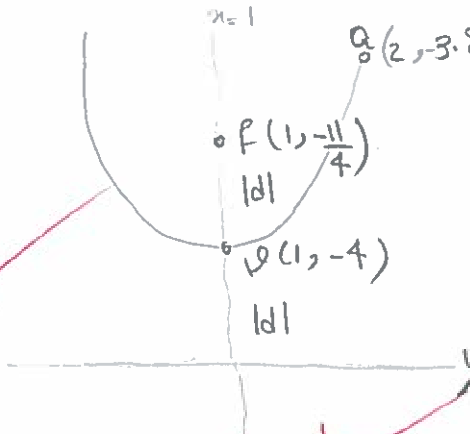
Q 8. Roughly, sketch $y = 0.2(x - 1)^2 - 4$.

(i) Find the focus point, say F.

$5(y + 4) = (x - 1)^2$

$4d = 5 \Rightarrow d = \frac{5}{4} > 0$

$V(1, -4)$ $\quad f\left(1, -4 + \frac{5}{4}\right) \Rightarrow f\left(1, \frac{-11}{4}\right)$



(ii) Find the directrix line.

directrix $\Rightarrow y = -4 - \frac{5}{4} = -\frac{21}{4} = -5.25$

(iii) Given $Q = (2, -3.8)$ lies on the curve. Find the distance between Q and F.

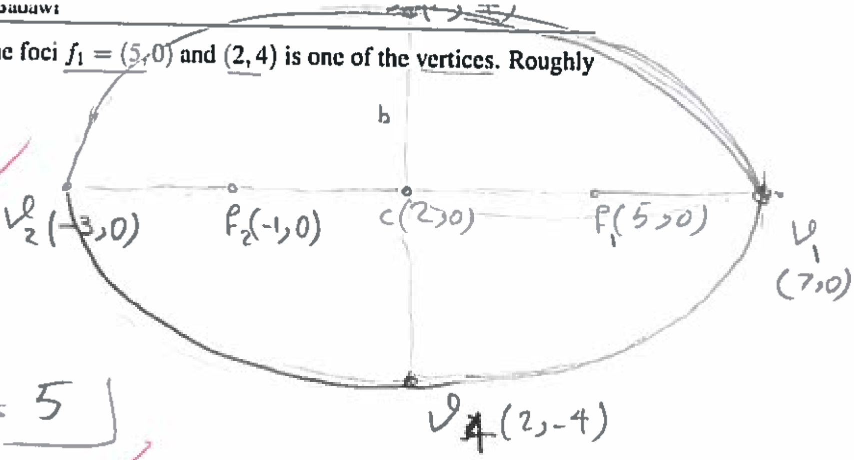
first way

$|QF| = \sqrt{(1 - 2)^2 + ((-2.75) + 3.8)^2} = \sqrt{1 + 1.1025} = \frac{29}{20} = 1.45$

$|QF| = |QL| = |-3.8 + 5.25| = 1.45$

second way

Q 9. Given an ellipse with center point $c = (2, 0)$, one of the foci $f_1 = (5, 0)$ and $(2, 4)$ is one of the vertices. Roughly sketch such ellipse.



(i) Find the second foci, f_2.

$$|cf_1| = 5 - 2 = 3$$

$$f_2 = (2-3, 0) = (-1, 0)$$

(ii) Find the ellipse-constant k.

$$a = \sqrt{b^2 + |cf_1|^2} = \sqrt{16+9} = 5$$

$$b = |cv_1| = 4 - 0 = 4$$

$$K = 2a = 2 \times 5 = 10$$

(iii) Find all 4 vertices (note that one is already given).

$$v_1\,(2+5, 0) = (7, 0)$$

$$v_2\,(2-5, 0) = (-3, 0)$$

$$v_3\,(2, 4)$$

$$v_4\,(2, 0-4) = (2, -4)$$

(iv) Write down the equation of the ellipse.

$$\frac{(x-2)^2}{25} + \frac{(y-0)^2}{16} = 1$$

Q 10. (i) Find the equation of the plane that contains the three points: $\underset{Q_1}{(0, 0, 1)}$, $\underset{Q_2}{(0, 1, 0)}$ and $\underset{Q_3}{(2, 4, -4)}$.

$$v = \overrightarrow{Q_1Q_2} = \langle 0, 1, -1 \rangle$$

$$\Rightarrow N = v \times U = \langle -1, -2, -2 \rangle$$

$$U = \overrightarrow{Q_1Q_3} = \langle 2, 4, -5 \rangle$$

$$Q_1(0,0,1) \longrightarrow P: -1(x-0) - 2(y-0) - 2(z+1) = 0$$

(ii) Find the distance between the point $Q = (1, 2, 4)$ and the plane $2x + 2y - z + 7 = 0$.

$$|QL| = \frac{|2 + 4 - 4 + 7|}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{9}{\sqrt{4+4+1}} = \frac{9}{3} = 3$$

(iii) Can we draw the vector $3i + 2j + 7k$ inside the plane $-5x + 2y - 3z = 0$? explain.

$$v = \langle 3, 2, 7 \rangle$$

a point on plane $\longrightarrow$ $x = 0,\ y = 0 \Rightarrow 0 + 0 - 3z = 0 \Rightarrow z = 0 \Rightarrow Q(0,0,0)$

Q is initial point $\longrightarrow$ $Q_2 = (3, 2, 7) \rightarrow$ terminal Point, we should check terminal point is also on the plane or not

$$-5(3) + 2(2) - 3(7) = -15 + 4 - 21 \neq 0$$ $\rightarrow$ it's not on plane so we can't draw v on the plane

Q 11. (i) Does the line $L: x = t+1, y = 2t-3, z = 5t$ intersect the line $M: x = s, y = s-1, z = 3s+3$? if yes find the intersection point.

$t+1 = s \Rightarrow t - s = -1 \Longrightarrow t = 3, s = 4$

$2t-3 = s-1 \Rightarrow 2t - s = 2$

$z = 5t = 5(3) = 15$ it intersect $\begin{cases} x = t+1 = 4 \\ y = 2t-3 = 6-3 = 3 \\ z = 15 \end{cases}$

$z = 3s+3 = 3(4)+3 = 15$ ($3(4) = 12$)

intersect point $= Q = (4, 3, 15)$

(ii) Find the area of the triangle with vertices: (4, 0), (0, 6), (8, 6).

A B C

$A'(4,0,0)$
$B'(0,6,0) \Longrightarrow$
$C'(8,6,0)$

$v = \overrightarrow{A'B'} = \langle -4, 6, 0\rangle \Longrightarrow |U \times v| = 48$

$U = \overrightarrow{A'C'} = \langle 4, 6, 0\rangle$

$S = \frac{1}{2}|U \times v| = \frac{48}{2} = 24$

(iii) Let $u = \langle 1, 6\rangle$, $v = \langle 3, 4\rangle$. Find $proj_v^u$ and $|proj_v^u|$.

$U = \langle 1, 6\rangle$, $v = \langle 3, 4\rangle$

$\text{Proj}_v^U = \frac{U \cdot v}{|v|^2} v = \frac{27}{25}\langle 3, 4\rangle = \langle \frac{81}{25}, \frac{108}{25}\rangle = \langle 3.24, 4.3$

$U \cdot v = 3 + 24 = 27$

$|v| = \sqrt{9+16} = 5$

$|\text{Proj}_v^U| = \frac{U \cdot v}{|v|} = \frac{27}{5} = 5.4$

(iv) The two planes: $x + y + z - 10 = 0$ and $2x + 2y + 3z - 21 = 0$ intersect in a line L. Find the parametric equations of L.

$N_1 = \langle 1, 1, 1\rangle$

$\Longrightarrow v = N_1 \times N_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \langle 1, -1, 0\rangle$

$N_2 = \langle 2, 2, 3\rangle$

$x = 0 \quad \begin{cases} y + z = 10 \\ 2y + 3z = 21 \end{cases} \Longrightarrow y = 9, z = 1, x = 0 \Rightarrow Q(0, 9, 1)$

equations $L: (0, 9, 1) + vt = (0, 9, 1) + \langle 1, -1, 0\rangle t$

$L: (0 + t, 9 - t, 1) \Rightarrow \begin{cases} x = t \\ y = 9 - t \\ z = 1 \end{cases} \quad (t \in \mathbb{R})$

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com